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Closed generalized elastic curves in $\mathbf{S}^2(1)$

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Abstract

We study the existence and stability of (closed) curves in $\mathbf{S}^2(1)$ which are critical points of generic curvature energy functionals. Firstly, we compute the first and second variation formulas and obtain first integrals of the Euler–Lagrange equations, then we establish conditions under which critical points close up. We apply the results to analyze two concrete situations: a natural generalization of the classical Euler–Bernoulli elastic functional and a constrained version of the total curvature functional.

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1. Introduction

The curvature κ, of a given curve $\gamma : \mathbf{I} \to \mathbf{M}^n$ in a Riemannian manifold, can be interpreted as the tension that γ receives at each point as a result of the way it is immersed in the surrounding space. Bernoulli proposed in 1740 a simple geometric model for an *elastic curve* in $\mathbf{R}^2$, according to which an *elastic curve* or *elastica* is a critical point of the *elastic energy* functional $\int_\gamma \kappa^2$. Elastic curves in $\mathbf{R}^2$ were already classified by Euler in 1743 but it was not until 1928 that they were studied also in $\mathbf{R}^3$ by Radon [19], who derived the Euler–Lagrange equations and showed that they can be integrated by quadratures. The elastica problem in real space forms has been recently considered under different approaches (see for instance [7,11–14]).

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More generally, for a given differentiable function $P(\kappa)$, the geometric importance of minimizing a *curvature energy* functional of the type $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$, defined on a certain space of curves in the three-dimensional Euclidean space $\mathbf{R}^3$, was pointed out by Blaschke in his book on Differential Geometry [5], where it is refereed to as *Radon's problem*. On the other hand, actions defined by $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ in constant curvature space–times have been used to describe models of relativistic particle with κ being the proper acceleration of the particle [8,15,16,18].

Our purpose here is to study critical points of curvature energy functionals in the standard 2-sphere paying special attention to the closed ones. Apart from their own geometric significance, closed critical points of $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ have been used also to provide construction methods of Chen–Willmore submanifolds in higher dimensional spheres [2,6,17].

First part of the paper develops the common starting point for the various particular cases that will be explored at the last sections. In Section 2 we obtain the Euler–Lagrange equation of $\mathbf{\Theta}$, and briefly derive its first integral, since they are basically known in the literature. Guided by the ideas of Langer and Singer [12], we use a coordinate system adapted to the problem, to give in Section 3 conditions to be satisfied by critical points with periodic curvature in order to close up. In Section 4 we obtain an expression for the second variation formula which allows us to get some applications to the stability of constant curvature critical points.

The two final sections are devoted to apply the previous results to study natural choices of concrete energy functionals. We choose, two particularly interesting concrete situations. In both cases, we solve the Euler–Lagrange equation obtaining periodic solutions and give an explicit description of the closure condition by using the Jacobi elliptic functions.

The simplest choice for $P(\kappa)$ is $P(\kappa) = \kappa^r$, $r \in \mathbf{N} \cup \{0\}$, and we refer to the critical points of $\mathbf{\Theta}$ as *generalized r-elastica*. If $r = 0$, then $P(\kappa)$ is constant and *generalized 0-elastica* of $\mathbf{\Theta}$ are simply geodesics. If $r = 1$, we have that $\mathbf{\Theta}$ is the *total curvature functional*, and it has been studied in [3,4]. In particular, there are no closed critical points of the *total curvature* in the 2-sphere. When $r = 2$, we have the classical *elastic curves*. Closed elastic curves in the 2-sphere have been classified by Langer and Singer [12]. We show that, for $r > 2$, there are no closed *generalized r-elastica* in $\mathbf{S}^2(G)$ (the 2-sphere of curvature G), other than geodesics. This surprising result forces us to enlarge the class of numbers where r moves if we expect to find non-trivial closed critical points. If we want to find periodic solutions of the Euler–Lagrange equation we prove r must be lower than the curvature of the sphere G. Assuming $G = 1$, and $r = 1/2$, we show that closed critical curves in $\mathbf{S}^2(1)$ form a countably infinite family.

On the other hand, it is known, that plane curves are the critical points of the *total curvature* functional in $\mathbf{R}^3$ [3]. It is an interesting question whether or not other critical points in $\mathbf{R}^3$ of this functional appear when we take variations constrained to the sphere. This leads us to consider in Section 6 functionals of the type $\mathcal{F}^\lambda(\gamma) = \int_\gamma (\kappa^2 + \lambda)^{1/2}\,\mathrm{d}s$, $\lambda > 0$, defined on spaces of curves in $\mathbf{S}^2(G)$. This type of functionals have also appeared as models of a relativistic particle with maximal proper acceleration [8,16]. If $\lambda > G$ the only closed critical points are geodesics. If $\lambda = G$, we have the *total* $\mathbf{R}^3$*-curvature* constrained to $\mathbf{S}^2(G)$. In this case circles are the only closed critical points as expected. However, we prove that there is a family of generalized helices in $\mathbf{S}^2(1)$ which are critical points for the problem with pinned ends and first boundary data. If $\lambda < G$, we show that there exist a countably infinite family of closed critical points and determine the multiple covers of the geodesics that are stable.

2. Preliminaries

We shall denote by $\mathcal{D}$ the space of regular C^4-curves in $\mathbf{S}^2(G)$, that is the space of C^4-immersions of $\mathbf{I} = [0, 1]$ in $\mathbf{S}^2(G)$,

$$\mathcal{D} = \left\{\gamma : \mathbf{I} \to \mathbf{S}^2(G);\ \gamma \in C^4(\mathbf{I});\ \frac{\mathrm{d}\gamma}{\mathrm{d}t} \neq 0\right\}.$$

For a given curve γ in $\mathcal{D}$, we denote by $V(t) = \mathrm{d}\gamma/\mathrm{d}t = \gamma'$ the tangent vector field; $T(t)$ denotes unit tangent vector; $v(t) = \langle V, V\rangle^{1/2}$ the speed of γ; $N(t)$ the unit normal to γ; and κ denotes the oriented geodesic curvature of γ in $\mathbf{S}^2(G)$.

We take $P(t)$ a C^∞ function and consider the following *curvature energy functional*:

$$\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa) = \int_0^L P(\kappa)\,\mathrm{d}s = \int_0^1 P(\kappa) v\,\mathrm{d}t \tag{1}$$

acting on $\mathcal{D}$.

Let us assume that $\Gamma(w, t) = \gamma_w(t) : (-\varepsilon, \varepsilon)\times \to \mathbf{S}^2(G)$ is a variation of γ in $\mathcal{D}$ with $\gamma(0, t) = \gamma$, whose variation vector field along the curve γ is $W = W(t) = (\partial\Gamma/\partial w)(0, t)$. We shall also write $V = V(w, t)$, $W = W(w, t)$, $T = T(w, t)$, $v = v(w, t)$, $N = N(w, t)$, etc., with the obvious meanings. As usual we use s to denote the arclength parameter and $\gamma(s)$, $V = V(w, s)$, $W = W(w, s)$, $T = T(w, s)$, $v = v(w, s)$, etc., for the corresponding reparameterizations. Then $s \in [0, L]$, where L is the length of γ.

The restriction of a *curvature energy functional* to a variation is denoted by the same letter, $\mathbf{\Theta}(w) = \mathbf{\Theta}(\gamma_w(t))$. Now we want to compute the first derivative of $\mathbf{\Theta}(w)$.

We shall make use of the following notation: $P'(\kappa) = \mathrm{d}P/\mathrm{d}\kappa$, and

$$\mathcal{K} = P'(\kappa)N, \qquad \mathcal{J} = (\kappa P'(\kappa) - P(\kappa))T + \frac{\mathrm{d}P'}{\mathrm{d}s}N,$$
$$\mathcal{E} = \left((\kappa^2 + G)P'(\kappa) + \frac{\mathrm{d}^2 P'}{\mathrm{d}s^2} - \kappa P(\kappa)\right) N. \tag{2}$$

By using Lemma 1 of [12], the first *Frenet formula* $\nabla_T T = \kappa N$ and integrating by parts, one can obtain the following proposition.

Proposition 1 (First variation formula). *Let $\Gamma(w, s) = \gamma_w(t)$ be a variation of γ by curves in $\mathcal{D}$ and $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ a curvature energy functional acting on $\mathcal{D}$. Then the following formula holds*:

$$\left.\frac{\mathrm{d}\mathbf{\Theta}}{\mathrm{d}w}\right|_{w=0} = \left(\int_0^L \langle \mathcal{E}, W\rangle\,\mathrm{d}s\right) + \mathcal{B}[W, \gamma]_0^L, \tag{3}$$

where the boundary term is given by $\mathcal{B}[W, \gamma]_0^L = [\langle \mathcal{K}, \nabla_T W\rangle - \langle \mathcal{J}, W\rangle]_0^L$.

Now, we may see $\mathbf{\Theta}$ acting on subspaces of $\mathcal{D}$ formed by curves which satisfy, in addition, a suitable set of boundary conditions. With an eye in our applications, we first consider that $\mathbf{\Theta}$ acts on Ω, the space of smooth closed curves of $\mathbf{S}^2(G)$ (although the following computations might be equally apply to other cases, for instance, to the space of curves Ω_{pq}, with pinned ends and given first order boundary data). In such cases, the above boundary term drops out. Thus a critical point of $\mathbf{\Theta}$ in such spaces will be characterized by the *Euler–Lagrange equation* $\mathcal{E} = 0$, in other words by

$$(\kappa^2 + G)P'(\kappa) + \frac{\mathrm{d}^2 P'}{\mathrm{d}s^2} = \kappa P(\kappa). \tag{4}$$

One may want to consult [10] for a different derivation of (4).

If $\mathrm{d}P'/\mathrm{d}s = 0$, then we may assume $P(\kappa) = \kappa + \lambda$ and either we do not have critical points if $\lambda = 0$, or the only closed critical points are circles of curvature G/λ otherwise [3]. This case is basically the *total curvature* functional.

Assume $\mathrm{d}P'/\mathrm{d}s \neq 0$. Then $\mathcal{E} = 0$ on a critical point γ of $\mathbf{\Theta}$. To facilitate integration of the Euler–Lagrange equations we look for a first integral of it. From (2), we observe that we may write $\mathcal{E} = \nabla_T \mathcal{J} + G\mathcal{K}$, hence

$$0 = \langle \mathcal{E}, \mathcal{J} \rangle = \langle \nabla_T \mathcal{J} + G\mathcal{K}, \mathcal{J} \rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(\langle \mathcal{J}, \mathcal{J} \rangle + G\langle \mathcal{K}, \mathcal{K} \rangle)(s),$$

and therefore $\langle \mathcal{J}, \mathcal{J} \rangle + G\langle \mathcal{K}, \mathcal{K} \rangle$ is constant along γ, which in combination with (2) allows us to obtain a first integral of $\mathcal{E} = 0$

$$\left(\frac{\mathrm{d}P'}{\mathrm{d}s}\right)^2 + (\kappa P'(\kappa) - P(\kappa))^2 + G(P'(\kappa))^2 = d. \tag{5}$$

First integrals of the Euler–Lagrange equations (4) in space forms are basically known [1,10,16].

3. Closure conditions

Since we are mainly interested in closed solutions of the Euler–Lagrange equations, we must seek for periodic solutions of (5). Any periodic solution will give rise to a unique (up to isometries) curve in $\mathbf{S}^2(G)$ which need not be a closed curve. On the other hand, in their study of the classical elastica in two-dimensional space forms [12], Langer and Singer find *Killing fields* along an elastic curve $\gamma(s)$ expressible in terms of the local invariants of the curve and use them to determine closure conditions. By using this approach we can also establish closure condition for critical curves with periodic curvature in the general case [1].

Let $\gamma : \mathbf{I} = [0, 1] \to \mathbf{S}^2(G)$ be an immersed regular curve in the 2-sphere of constant sectional curvature G. A vector field W is called a *Killing field* along γ [12], if for any variation in the direction of W we have

$$\frac{\partial v}{\partial w} = \frac{\partial \kappa}{\partial w} = 0. \tag{6}$$

From Lemma 1 of [12], we can see that W is a Killing field along γ, if and only if,

$$\langle \nabla_T W, T\rangle = 0, \tag{7}$$

$$\langle \nabla_T^2 W, N\rangle + G\langle W, N\rangle = 0. \tag{8}$$

Moreover, it was proved in [12] that a Killing field along a curve γ contained in a two-dimensional real space form is the restriction to γ of a Killing field defined on whole space. Then we have the following proposition.

Proposition 2. *Let $\gamma : \mathbf{I} = [0, 1] \to \mathbf{S}^2(G)$ be an immersed regular curve in $\mathbf{S}^2(G)$ which is a critical point of $\mathbf{\Theta}$ acting on $\mathcal{D}$. Then the vector field $\mathcal{J}$ defined as in (2) is a Killing field along γ and, therefore, it is the restriction to γ of a Killing field of $\mathbf{S}^2(G)$.*

Proof. Using (2), we have $\nabla_T \mathcal{J} = ((\mathrm{d}^2 P'/\mathrm{d}s^2) + (\kappa P'(\kappa) - P(\kappa)))N$ which trivially satisfies (7). Differentiating again and using the Euler–Lagrange equation (4), we have $\langle \nabla_T^2 \mathcal{J}, N\rangle = -G(\mathrm{d}P'/\mathrm{d}s)$ which shows (8). □

Assume that κ is a periodic solution of (5) with period ρ. Let us denote by γ the curve in $\mathbf{S}^2(G)$ of curvature κ. Now, one can use the above proposition to choose spherical coordinates $x(\theta, \psi) = (\cos\theta \sin\psi,\ \sin\theta\sin\psi,\ \cos\psi)$, so that its equator gives the only integral geodesic of $\mathcal{J} : x_\theta = b\mathcal{J}$. Combining this with (2) and (5), we have

$$\theta'(s) = \frac{\langle T, x_\theta\rangle}{\sin^2\psi} = \frac{\kappa P'(\kappa) - P(\kappa)}{b(d - G(P'(\kappa))^2)}. \tag{9}$$

Hence we have the following proposition.

Proposition 3. *Let κ be a periodic solution of (5) with period ρ. Let γ be the corresponding curve in $\mathbf{S}^2(G)$. Then γ is closed (and therefore a critical point of $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ in Ω), if and only if, its progression angle in one period of the curvature,*

$$\Lambda^{\mathbf{\Theta}} = \int_0^\rho \frac{\kappa P'(\kappa) - P(\kappa)}{b(d - G(P'(\kappa))^2)}\,\mathrm{d}s, \tag{10}$$

is a rational multiple of 2π.

Remarks. In many applications it is also important to determine the progression angle range of variation as accurately as possible [2,12]. $\Lambda^{\mathbf{\Theta}}$ depends on two parameters $d \in \mathbf{R}^+$, $b \in \mathbf{R}$. When looking for periodic solutions of (5), d is normally restricted to lie in a certain interval C. Then, if we want to know the variation of $\Lambda^{\mathbf{\Theta}}$ as d moves in C, we must get some control on the normalization factor b. This is possible in many cases. Let d be a given number in C and denote by κ_d a periodic solution of (5) and γ_d the curve with curvature κ_d in $\mathbf{S}^2(G)$.

(1) Assume first that κ_d reaches a zero of P' at some point s_0 and that γ_d crosses the equator determined by the rotation field $\mathcal{J}$. Since $P'(\kappa(s_0)) = P'(s_0) = 0$, we have that $(P')^2$ takes an absolute minimum in $\kappa(s_0)$ and, therefore $|\mathcal{J}|^2 = d - G(P')^2$ takes its maximum value on γ_d at s_0, in other words γ_d crosses the equator at the instant s_0, and

then $\psi(s_0) = \pi/2$. Hence evaluation at s_0 of $|x_\theta|^2 = b^2|\mathcal{J}|^2 = \sin^2\psi/G$, would give $Gb^2 d = 1$, which can be used to simplify (10).

(2) As another example, assume now that γ_d has a vertex at s_0 ($\kappa_d(s_0) = 0$), which is not either of the poles of $\mathbf{S}^2(G)$ as determined by $\mathcal{J}$: $(\kappa_d P'(\kappa_d) - P(\kappa_d))(s_0) \neq 0$. Let Σ be the integral curve of $\mathcal{J}$ at $\gamma_d(s_0) = p_0$ and denote by κ_0 the geodesic curvature of Σ at p_0. Then $\kappa_0 = -GP'/(\kappa_d P'(\kappa_d) - P(\kappa_d))(s_0)$, which combined with (2) and (5) gives $|x_\theta|^2 = b^2|\mathcal{J}|^2 = b^2(\kappa_d P'(\kappa_d) - P(\kappa_d))^2 = 1/(\kappa_0^2 + G) = (\kappa_d P'(\kappa_d) - P(\kappa_d))^2/Gd$. Therefore, we get $Gb^2 d = 1$ again.

4. Second variation formula

Let us compute now the second variation formula. Since our applications will be focused mainly in closed critical curves of the sphere, we restrict ourselves to Ω the space of closed curves in $\mathbf{S}^2(G)$. Assume that γ is a critical point of $\mathbf{\Theta} : \Omega \to \mathbf{R}$, $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ in the real space form $\mathbf{S}^2(G)$. For any variation of γ in Ω, $\Gamma(w, s)$, the first derivative is $\mathrm{d}\mathbf{\Theta}/\mathrm{d}w = \int_{\gamma_w} \langle \mathcal{E}, W\rangle$ and, therefore, since $\mathcal{E}(\gamma) = 0$, we can write

$$\left.\frac{\mathrm{d}^2\mathbf{\Theta}}{\mathrm{d}w^2}\right|_{w=0} = \int_\gamma \langle W, \nabla_W \mathcal{E}\rangle. \tag{11}$$

We may assume that $W(s) = \phi(s)N(s)$, then

$$\left.\frac{\mathrm{d}^2\mathbf{\Theta}}{\mathrm{d}w^2}\right|_{w=0} = \int_\gamma \phi W\langle N, \mathcal{E}\rangle. \tag{12}$$

Now

$$W\langle \mathcal{E}, N\rangle = (\kappa_s^2 P'''' + \kappa_{ss} P''' + (\kappa^2 + G)P'' + \kappa P' - P)W(\kappa) + 2\kappa_s P''' W(\kappa_s) + P'' W(\kappa_{ss}). \tag{13}$$

But using formula (5) of Lemma 1.1 [12], we have

$$W(\kappa) = \phi_{ss} + (\kappa^2 + G)\phi. \tag{14}$$

On the other hand, by formula (2) of the same lemma, $[W, T] = -\langle \nabla_T W, T\rangle T = \kappa\phi T$. Then, $W(\kappa_s) = W(T(\kappa)) = [W, T](\kappa) + T(W(\kappa)) = \phi\kappa\kappa_s + (W(\kappa))_s$, obtaining thus

$$W(\kappa_s) = \phi_{sss} + (\kappa^2 + G)\phi_s + 3\kappa\kappa_s\phi. \tag{15}$$

Analogously, $W(\kappa_{ss}) = \phi\kappa\kappa_{ss} + (W(\kappa_s))_s$ would give

$$W(\kappa_{ss}) = \phi_{ssss} + (\kappa^2 + G)\phi_{ss} + 5\kappa\kappa_s\phi_s + (3\kappa_s^2 + 4\kappa\kappa_{ss})\phi. \tag{16}$$

Hence combining (12)–(16) and integrating by parts, we obtain after a long computation the following proposition.

Proposition 4. *Assume that γ is a critical point with curvature κ of the curvature energy functional $\mathbf{\Theta} : \Omega \to \mathbf{R}$, $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ acting on the space of closed curves of the sphere $\mathbf{S}^2(G)$, then*

$$\left.\frac{\mathrm{d}^2\mathbf{\Theta}}{\mathrm{d}w^2}\right|_{w=0} = \int_\gamma m_1(s)\phi_{ss}^2 - \int_\gamma m_2(s)\phi_s^2 + \int_\gamma m_3(s)\phi^2, \tag{17}$$

where

$$m_1(s) = P''(\kappa), \qquad m_2(s) = 2(\kappa^2 + G)P''(\kappa) + \kappa P'(\kappa) - P(\kappa),$$
$$m_3(s) = (\kappa^2 + G)\frac{\mathrm{d}^2 P''}{\mathrm{d}s^2} + (\kappa^2)_s\frac{\mathrm{d}P''}{\mathrm{d}s} + (\kappa^2 + G)^2 P''(\kappa) + 4\kappa\frac{\mathrm{d}^2 P'}{\mathrm{d}s^2} + 3\kappa_s\frac{\mathrm{d}P'}{\mathrm{d}s}$$
$$+ \kappa(\kappa^2 + G)P'(\kappa) - (\kappa^2 + G)P(\kappa). \tag{18}$$

In many applications, circles are critical points of energy functionals. For instance, if P is an even function then geodesics are always critical points. In such cases second variation formula simplifies to the following proposition.

Proposition 5. *Let γ be a circle with constant curvature $\kappa \equiv \alpha$ of a sphere $\mathbf{S}^2(G)$. Assume that γ is a critical point of the energy functional $\mathbf{\Theta} : \Omega \to \mathbf{R}$, $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$ acting on closed curves of the sphere $\mathbf{S}^2(G)$. Then the second variation formula at γ is*

$$\left.\frac{\mathrm{d}^2\mathbf{\Theta}}{\mathrm{d}w^2}\right|_{w=0} = \int_\gamma \phi_{ss}^2 P''(\alpha) - \int_\gamma (2(\alpha^2 + G)P''(\alpha) + \alpha P'(\alpha) - P(\alpha))\phi_s^2$$
$$+ \int_\gamma (\alpha^2 + G)((\alpha^2 + G)P''(\alpha) + \alpha P'(\alpha) - P(\alpha))\phi^2. \tag{19}$$

Let us denote by γ^m the m-cover of the above circle γ. We parameterize it as $\gamma^m(s) = (r\cos(s/r), r\sin(s/r), 0)$, $s \in [0, L = 2\pi mr]$, where $r = (\alpha^2 + G)^{-1/2}$ is the radius of γ. Then we can write ϕ as a Fourier series, $\phi(s) = (a_0/2) + \sum_{h=1}^{\infty}[a_h\cos(h(s/mr)) + b_h\sin(h(s/mr))]$, which can be put in (19).

If $P''(\alpha) \neq 0$ we get

$$\left.\frac{\mathrm{d}^2\mathbf{\Theta}}{\mathrm{d}w^2}\right|_{w=0} = \pi m(\alpha^2 + G)^{3/2}P''(\alpha)(1 - \zeta)\frac{a_0^2}{2} + \pi m(\alpha^2 + G)^{3/2}P''(\alpha)$$
$$\times \sum_{h=1}^{\infty}\left[(a_h^2 + b_h^2)\left(\left(\frac{h}{m}\right)^2 - 1\right)\left(\left(\frac{h}{m}\right)^2 - 1 + \zeta\right)\right], \tag{20}$$

where

$$\zeta = \frac{P(\alpha)G}{(\alpha^2 + G)^2 P''(\alpha)}. \tag{21}$$

If $P''(\alpha) = 0$, then

$$\left.\frac{\mathrm{d}^2\mathbf{\Theta}}{\mathrm{d}w^2}\right|_{w=0} = -P(\alpha)G\left\{\pi m(\alpha^2 + G)^{-1/2}\frac{a_0^2}{2} + \sum_{h=1}^{+\infty}\left[(a_h^2 + b_h^2)\left(1 - \left(\frac{h}{m}\right)^2\right)\right]\right\}. \tag{22}$$

In our context, we shall say that a critical point γ of $\mathbf{\Theta}$ is *stable*, if for any variation γ_w of γ, $(\mathrm{d}^2\mathbf{\Theta}/\mathrm{d}w^2)|_{w=0} \geq 0$. By inspection of the above formulas (20) and (22), we can draw some consequences about the stability of circles in spheres. For instance, if $P''(\alpha) = 0$ and $P(\alpha) \neq 0$, then we see from (22) that there are choices of the variation field W which makes negative the second derivative. Proceeding in a similar way we obtain the following proposition.

Proposition 6. *Let $\mathbf{\Theta} : \Omega \to \mathbf{R}$, $\mathbf{\Theta}(\gamma) = \int_\gamma P(\kappa)$, a curvature energy functional acting on the space of closed curves of $\mathbf{S}^2(G)$. Assume that a circle γ of curvature α is a critical point of $\mathbf{\Theta}$ and denote by γ^m the m-cover of γ. Then γ^m is stable, if and only if, $P''(\alpha) > 0$ and*

$$\left| m\left(1 - \sqrt{1 - \frac{GP(\alpha)}{P''(\alpha)(\alpha^2 + G)^2}}\right)\right| \leq 1. \tag{23}$$

In particular, if γ is a geodesic, we have the following corollary.

Corollary 7. *Under the conditions of the above proposition, suppose that γ is a geodesic in $\mathbf{S}^2(G)$. Then γ^m is stable, if and only if, $P''(\alpha) > 0$ and*

$$\left| m\left(1 - \sqrt{1 - \frac{P(0)}{GP''(0)}}\right)\right| \leq 1. \tag{24}$$

Condition (24) is equivalent to

$$-G\frac{1}{m}\left(2 + \frac{1}{m}\right) \leq \frac{P(0)}{P''(0)} \leq G\frac{1}{m}\left(2 - \frac{1}{m}\right). \tag{25}$$

5. Generalized elastic curves

The most natural choice for $P(\kappa)$ is $P(\kappa) = \kappa^r$, $r \in \mathbf{N}$, and $\mathbf{\Theta}(\gamma) = \int_\gamma \kappa^r$. If $r = 1$, it is the total curvature functional and one can easily see from (4) that there are no solutions of the Euler–Lagrange equations. If $r = 2$, $\mathbf{\Theta}(\gamma) = \int_\gamma \kappa^2$ is the classical *Euler–Bernoulli elastic* functional whose closed critical points in $\mathbf{S}^2(1)$ have been studied and classified by Langer and Singer [12]. Now assume that r is a natural number greater than 2 and, without loss of generality, that $G = 1$. If κ is constant, then from (4) we have $\kappa^{r-1}((r-1)\kappa^2 + r) = 0$ and then there are no critical points of constant curvature other than geodesics. If κ were not a constant function, then we could use the first integral of the Euler–Lagrange equation (5) to get

$$r^2(r-1)^2\kappa^{2(r-2)}\kappa_s^2 = d - (r-1)^2\kappa^{2r} - r^2\kappa^{2r-2}. \tag{26}$$

The curvature κ of a closed critical point of $\mathbf{\Theta}$ should be a periodic solution of (26). Then $\kappa(s) \in [\beta, \alpha]$, where β and α are the minimum and maximum of $\kappa(s)$, respectively, what would imply that they are roots of the polynomial $Q(x) = d - (r-1)^2x^{2r} -$

r^2x^{2r-2}, $d > 0$. But then β should be positive, otherwise there would exist a point s_0 with $\kappa(s_0) = 0$ and we would have from (26) that $d = 0$, which is impossible. This is a contradiction since $Q(x)$ has only one positive root. Hence we have the following proposition.

Proposition 8. *Let* $\mathbf{\Theta} : \Omega \to \mathbf{R}$, $\mathbf{\Theta}(\gamma) = \int_\gamma \kappa^r$, $r \in \mathbf{N}$, *a curvature energy functional acting on the space of closed curves of* $\mathbf{S}^2(1)$. *Then*:

1. *if* $r = 1$, *there are no critical points of* $\mathbf{\Theta}$ (*no matter if they are closed or not*);
2. *if* $r = 2$, $\mathbf{\Theta}$ *is the classical Euler–Bernoulli elastic functional and its closed critical points have been classified by Langer and Singer* [12];
3. *if* $r > 2$, *the only closed critical points of* $\mathbf{\Theta}$ *are the geodesics*.

Geodesics are absolute minima if $r \geq 2$ is even and from (19) we see that on a geodesic, the second derivative $(\mathrm{d}^2\mathbf{\Theta}/\mathrm{d}w^2)|_{w=0} = 2\int(\phi_{ss}+\phi)^2 \geq 0$ if $r = 2$ and vanishes identically if $r > 2$.

The surprising result obtained in the above proposition leads one to ask if, instead we might find non-trivial closed critical points when r is not a natural number. For simplicity, we assume that $r = 1/2$ and that $\mathbf{\Theta}$ acts on $\tilde{\Omega}$ the space of convex ($\kappa > 0$) closed curves of $\mathbf{S}^2(1)$. From the Euler–Lagrange equation (4) we see that the only solutions with constant curvature are the circles with $\kappa = \sqrt{r/(1-r)} = 1$. Assuming that the curvature is not constant, we have to look for periodic solutions of the first integral (5) which for $r = 1/2$ reduces to

$$\kappa_s^2 = -4\kappa^2(\kappa^2 - 4d\kappa + 1), \tag{27}$$

with $d > 0$. Actually periodicity condition imply $d > 1/2$. Hence $Q(x) = -4x^2(x^2 - 4dx + 1)$ has two positive solutions α and $1/\alpha$ with $\alpha > 1$. Combining (27) and formula 2.226 of [9], we obtain

$$\kappa_\alpha(s) = \frac{2\alpha(1+\alpha^2)}{(1+\alpha^2)^2 + (1-\alpha^2)^2\cos 2\rho(s) - 2\alpha(\alpha^2-1)\sin 2\rho(s)}, \tag{28}$$

where $\rho(s) = s - \arctan(1/\alpha)$, $d = (1+\alpha^2)/4\alpha$, $\alpha \in (1,\infty)$. Thus $\kappa_\alpha(s)$ is a periodic solution of (27) which reaches its minimum value at $\kappa_\alpha(0) = 1/\alpha$ and the maximum at $\kappa_\alpha(\pi/2) = \alpha$. Therefore, we have proved that there exists a 1-parameter family $\{\kappa_\alpha(s); \alpha \in (1,\infty)\}$ of periodic solutions of (27) which are given by (28). Let $\{\gamma_\alpha; \alpha \in (1,\infty)\}$ be the corresponding 1-parameter family of curves in the 2-sphere. These are our candidates to be closed critical points of $\mathbf{\Theta}$.

Now, we are in condition to apply remark (2) at the end of Section 3, and we may take $bd^2 = 1$ in Proposition 3. Hence, a curve of the above family γ_α will close up, if and only if, it satisfies the closure condition (10), that is if and only if, the angular progression in one period of κ_α, which we denote now by

$$\Lambda(\alpha) = -\int_0^\pi \left(\frac{\sqrt{d}\kappa_\alpha^{3/2}(s)}{4d\kappa_\alpha(s) - 1}\right)\mathrm{d}s, \tag{29}$$

is a rational multiple of 2π. By combining (27) and (29) and $d = (1+\alpha^2)/4\alpha$ one has

$$\Lambda(\alpha) = 2\int_{1/\alpha}^{\alpha} \frac{\sqrt{((1+\alpha^2)/4\alpha)}\kappa^{1/2}}{(1-((1+\alpha^2)/\alpha)\kappa)\sqrt{(\kappa-(1/\alpha))(\alpha-\kappa)}}\,\mathrm{d}\kappa,$$

which after some computations and using formulas 3.137-3 and 3.131-3 of [9] give

$$\Lambda(\alpha) = -2\sqrt{\frac{1-p^2}{2-p^2}}\left[(1-p^2)\Pi\left(\frac{\pi}{2}, \nu, p\right) + K(p)\right], \tag{30}$$

where $K(p)$, $\Pi(\pi/2, \nu, p)$ are the complete elliptic integrals of the first and the third kind, respectively, of modulus $p = \sqrt{(\alpha^2-1)}/\alpha$ and $\nu = p^2(2-p^2)$. Now, p moves in $(0, 1)$ as α varies in $(1, \infty)$, then, we have $\lim_{p\to 0} \Lambda(\alpha) = -\sqrt{2}\pi$ and $\lim_{p\to 1} \Lambda(\alpha) = -\pi$. Hence the angular progression in one period of γ_α, $\Lambda(\alpha)$ increases continuously from $-\sqrt{2}\pi$ to $-\pi$ as α varies from 1 to ∞. Hence, we have proved the following proposition.

Proposition 9. *For any couple of integers* m, n *satisfying* $n < 2m < \sqrt{2}n$, *there exists a convex curve* $\gamma_{m,n}$, *which is a closed critical point of* $\mathbf{\Theta}(\gamma) = \int_\gamma \kappa^{1/2}$ *in the unit sphere* $\mathbf{S}^2(1)$. $\gamma_{m,n}$ *closes up after n periods of its curvature* (*given in* (28)) *and m trips around the equator* (*see* Fig. 1). *Any closed generalized* (1/2)*-elastica is obtained as above.*

To show that a pair of integers (m, n) determines the *generalized* (1/2)*-elastica* uniquely would require to show monotonicity of $\Lambda(\alpha)$ along $(1, \infty)$. We have established this numerically as part a of Fig. 2 shows.

Finally, let us denote ε_1^m the m-cover of the circle of geodesic curvature 1. It is the only circle of the sphere that is a critical point for $\mathbf{\Theta}(\gamma) = \int_\gamma \kappa^{1/2}$. By using (20) and (21) we see that ε_1^m is unstable. Actually, if $|m| = 1, 2$, then $(\mathrm{d}^2\mathbf{\Theta}/\mathrm{d}w^2)|_{w=0} < 0$, and they are "local maxima".

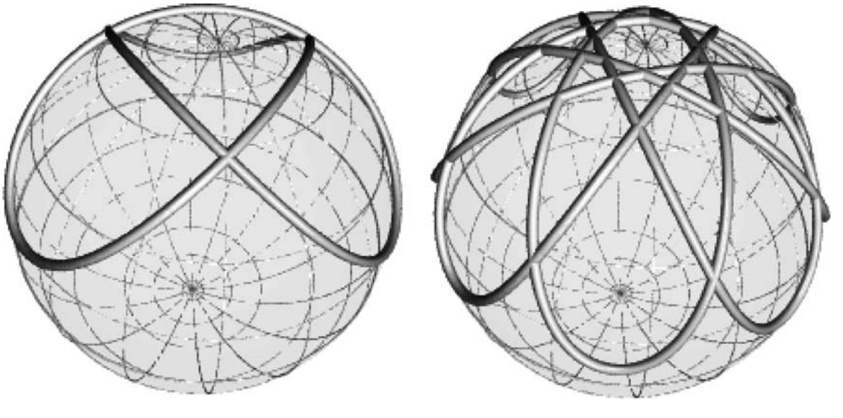

Fig. 1. Curves $\gamma_{2,3}$ and $\gamma_{5,8}$ closed critical points of $\int_\gamma \kappa^{1/2}$.

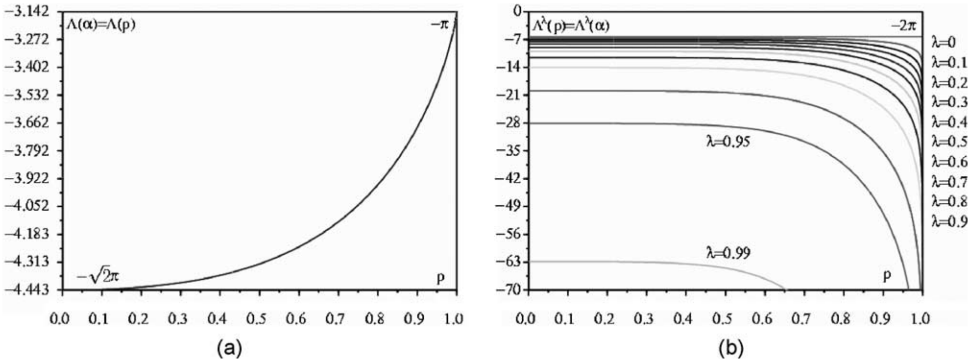

Fig. 2. (a) $P(\kappa) = \kappa^{1/2}$; (b) $P(\kappa) = (\kappa + \lambda)^r$.

6. Total R^3-curvature type functionals

Let $\mathbf{S}^2(G)$ be the two-dimensional unit sphere of constant Gaussian curvature G and $\gamma(s)$ an immersed curve in $\mathbf{S}^2(G)$ with geodesic curvature $\kappa(s)$, and curvature function in $\mathbf{R}^3$ denoted by $\bar\kappa(s)$. We have seen that there are no closed critical points in $\mathbf{S}^2(G)$ of the *total curvature* functional $\int \kappa$. On the other hand, plane curves are precisely the critical points of the *total curvature* in $\mathbf{R}^3$, $\int \bar\kappa$. It is natural then to investigate the existence of the closed critical points of $\int \bar\kappa$ when restricted to the sphere (other than circles). Thus, we consider, in a little more general setting, functionals of the following type:

$$\mathcal{F}^\lambda(\gamma) = \int_\gamma (\kappa^2 + \lambda)^{1/2}\, \mathrm{d}s, \tag{31}$$

where $\lambda > 0$, acting on the space of immersed closed curves Ω in $\mathbf{S}^2(G)$.

We take a variation of $\gamma(s)$ within the specified space of curves and use the first variation formula given in (3). In particular, $\gamma \in \Omega$ is a critical point of $\mathcal{F}^\lambda$ if and only if the following Euler–Lagrange equation is satisfied:

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{\kappa}{(\kappa^2+\lambda)^{1/2}}\right) + \frac{\kappa(\kappa^2+G)}{(\kappa^2+\lambda)^{1/2}} - \kappa(\kappa^2+\lambda)^{1/2} = 0. \tag{32}$$

We first investigate the existence of closed critical points of constant curvature. The *Euler–Lagrange* equations (32), are trivially satisfied by geodesics for any $\lambda > 0$. If κ is a non-zero constant, the above equation reduce to $G - \lambda = 0$. Thus, if $\lambda \neq G$ we do not have any other critical points with non-zero constant curvature κ. If $G = \lambda$, then every circle is a critical point for this functional.

Assume now that κ is a non-constant function. From (5) we get the first integral of the *Euler–Lagrange* equations of $\mathcal{F}^\lambda$

$$\kappa_s^2(s) = \left(\frac{\kappa^2+\lambda}{\lambda}\right)^2 [(d-G)\kappa^2 + \lambda(d-\lambda)], \tag{33}$$

where κ_s is the derivative with respect to the arclength parameter s and d is a constant of integration. This implies

$$\lambda < d < G. \tag{34}$$

Fix any $\lambda \in (0, G)$. In order to find closed critical curves, we need (33) to have periodic solutions. For any d verifying (34) we have a periodic solution which we may take with initial condition $\kappa_d(0) = 0$. In fact, by using, for example formula 2.266 of [9], we see that it is a periodic function $\kappa_d^2(s)$ given by

$$\kappa_d^2(s) = \frac{2\lambda(\lambda+\alpha)}{(2\lambda+\alpha) - \alpha \sin(2\sqrt{G-\lambda}s - (\pi/2))} - \lambda, \tag{35}$$

where $\alpha = \lambda(d-\lambda)/(G-d) > 0$ varies in $(0,\infty)$ as d does in (λ, G). Minima and maxima of the above solutions are reached at $\kappa_d(-(\pi/2\sqrt{G-\lambda})) = -\sqrt{\alpha}$ and $\kappa_d(\pi/2\sqrt{G-\lambda}) = \sqrt{\alpha}$, respectively. We indistinctively use either $\kappa_d^2(s)$ or $\kappa_\alpha^2(s)$ to denote curvature. Therefore, we have proved that for any $\lambda \in (0, G)$, there exists a 1-parameter family $\{\kappa_\alpha(s); \alpha \in (0,\infty)\}$ of periodic solutions of (33) which are given by (35). Let $\{\gamma_\alpha^\lambda; \alpha \in (0,\infty)\}$ be the corresponding 1-parameter family of curves in the 2-sphere. These are our candidates to closed critical points of $\mathcal{F}^\lambda$.

For any $d \in (\lambda, G)$, we have from Proposition 2 that $\mathcal{J} = (-\lambda/(\kappa^2+\lambda)^{1/2})T + (-\lambda\kappa_s/(\kappa^2+\lambda)^{3/2})N$, is a Killing field on γ_d^λ. Thus, using the Euler–Lagrange equation (4) we have $\mathrm{d}|\mathcal{J}|^2/\mathrm{d}s = -2G\lambda(\kappa\kappa_s/(\kappa^2+\lambda)^2)$. Then γ_d^λ crosses the equator at the zeroes of its curvature where $|\mathcal{J}|$ is maximum. Since κ and κ_s do not vanish simultaneously, we see that at the points where κ_s is zero, we have $\mathrm{d}^2|\mathcal{J}|^2/\mathrm{d}s^2 = (2G\lambda\kappa^2/(\kappa^2+\lambda))(G-\lambda) > 0$ and $|\mathcal{J}|^2$ reaches its minima. In short, γ_d^λ crosses the equator at inflection points and reaches at the vertices the farthest points from the equator. Moreover, $\mathcal{J} = (-\lambda/(\kappa^2+\lambda)^{1/2})T$ is tangent to γ_d^λ at the vertices. One can use now, one of the remarks at the end of Section 3 to prove that $b^2dG = 1$. Substituting this in (10), we see that a curve of the above family γ_α^λ will close up if and only if the angular progression in one period of γ_α^λ, which we denote now by

$$\Lambda^\lambda(\alpha) = \sqrt{G}\int_0^{2(\pi/\sqrt{G-\lambda})} \left(\frac{\lambda\sqrt{d}(\kappa_\alpha^2(s)+\lambda)^{1/2}}{(\kappa_\alpha^2(s)(1-d)-\lambda d)}\right)\mathrm{d}s, \tag{36}$$

is a rational multiple of 2π. Without loss of generality, we assume that $G = 1$. By using (33) one has

$$\Lambda^\lambda(\alpha) = 4\int_0^{\sqrt{\alpha}} \frac{\lambda^2\sqrt{d}}{(\kappa_\alpha^2(1-d)-\lambda d)\sqrt{(1-d)(\alpha^2-\kappa_\alpha^2)(\kappa_\alpha^2+\lambda)}}\,\mathrm{d}\kappa,$$

which after some computations and using formula 3.137.5 of [9] becomes

$$\Lambda^\lambda(\alpha) = \frac{-4\sqrt{\lambda(1-\lambda)}(1-p^2)}{\sqrt{\lambda+p^2(1-\lambda)}}\left[\Pi\left(\frac{\pi}{2},\nu,p\right) + \frac{\lambda+p^2(1-\lambda)}{(1-\lambda)(1-p^2)}K(p)\right], \tag{37}$$

where $K(p)$, $\Pi(\pi/2,\nu,p)$ are the complete elliptic integrals of the first and the third kind, respectively, of modulus $p = \sqrt{\alpha/(\lambda+\alpha)}$ and $\nu = p^2/p^2(1-\lambda)+\lambda$. By using the Heuman

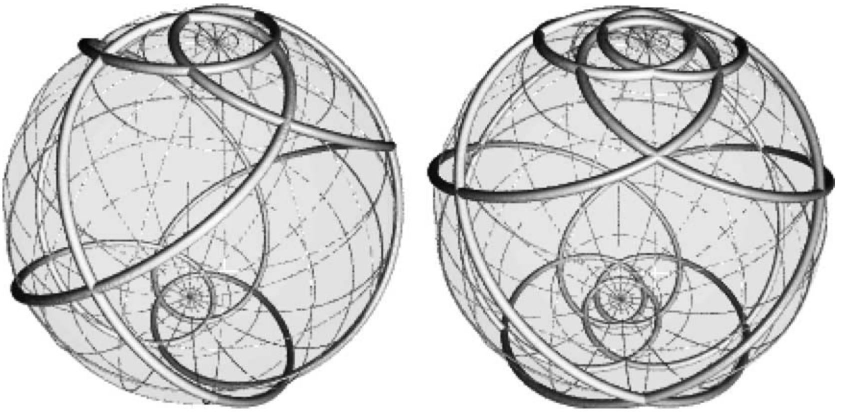

Fig. 3. Curves $\gamma_{7,3}^{0.5}$ and $\gamma_{10,3}^{0.7}$ closed critical points of $\int_\gamma(\kappa^2+\lambda)^{1/2}$.

Lambda function Λ_0, one can obtain

$$\Lambda^\lambda(\alpha) = \frac{-4\sqrt{\lambda}}{\sqrt{(1-\lambda)(\lambda+p^2(1-\lambda))}}K(p) - 2\pi(1-\Lambda_0(\vartheta, p)), \tag{38}$$

where $\vartheta = \arcsin\sqrt{\lambda/(\lambda+p^2(1-\lambda))}$, $\lambda \in (0,1)$. Now, p moves in $(0,1)$ as α varies in $(0,\infty)$. Then, we have $\lim_{p\to 0}\Lambda^\lambda(\alpha) = \lim_{\alpha\to 0}\Lambda^\lambda(\alpha) = -2(\pi/\sqrt{1-\lambda})$ and $\lim_{p\to 1}\Lambda^\lambda(\alpha) = \lim_{\alpha\to\infty}\Lambda^\lambda(\alpha) = -\infty$. Hence for any $\lambda \in (0,1)$ the angular progression of γ_α^λ, $\Lambda^\lambda(\alpha)$ decreases from $-2(\pi/\sqrt{1-\lambda})$ to $-\infty$ as α varies from 0 to ∞. Hence, we have proved the following proposition.

Proposition 10. *For any $\lambda \in (0,1)$, and for any couple of integers (m,n) satisfying $n < m\sqrt{1-\lambda}$, there exists a closed critical point $\gamma_{m,n}^\lambda$ of $\mathcal{F}^\lambda(\gamma) = \int_\gamma(\kappa^2+\lambda)^{1/2}\,\mathrm{d}s$ in $\mathbf{S}^2(1)$. $\gamma_{m,n}^\lambda$ closes up in n periods of its curvature (given in (35)) and m trips around the equator (see Fig. 3). Any closed critical point of $\mathcal{F}^\lambda$ is obtained in this way.*

Monotonicity of $\Lambda^\lambda(\alpha)$ would result in that the pair (m,n) determines uniquely $\gamma_{m,n}^\lambda$. We have checked this numerically, see Fig. 2b. Summarizing the above results, we get the following proposition.

Proposition 11. *Let $\mathcal{F}^\lambda : \Omega \to \mathbf{R}$ the energy functional defined in (31) acting on closed curves of $\mathbf{S}^2(G)$ the 2-sphere of curvature G. Then:*

1. *if $\lambda > 0$, the only closed critical points are the geodesics;*
2. *if $\lambda = G$, the only closed critical points are the circles;*
3. *if $0 < \lambda < G$, in addition to geodesics, the set of closed critical points of $\mathcal{F}^\lambda$ is a countably infinite family described in Proposition 10.*

We get now some consequences of the second variation formula and corollaries. Let us denote by ε_r^m, $m \in \mathbf{N}$, the m-cover of the circle of curvature r in $\mathbf{S}^2(G)$, then we have the following proposition.

Proposition 12. *Let $\mathcal{F}^\lambda : \Omega \to \mathbf{R}$, the energy functional defined in (31) acting on closed curves of $\mathbf{S^2}(G)$. Then*:

1. *If $0 < \lambda < G$ an m-geodesic ε_r^m is stable, if and only if*

$$\left(\frac{m-1}{m}\right)^2 \leq 1 - \frac{\lambda}{G}.$$

In particular, the 1-geodesics are stable for every λ; a 2-geodesic is stable if $\lambda < (3/4)G$; a 3-geodesic is stable if $\lambda < (7/16)G$ and so on.

2. *Assume that $\lambda = G$. Then ε_r^m is stable if and only if, $|m| = 1$. Actually, as a consequence of Fenchel's theorem, every 1-circle is an absolute minimum.*

As we see from Proposition 11, apart from the circles, there are no other closed critical points in the 2-sphere of the total $\mathbf{R}^3$-*curvature*. Of course one can obtain non-trivial critical points of this functional by modifying the boundary conditions. For instance, we may consider $\mathcal{F}(\gamma) = \int_\gamma \bar{\kappa} = \int_\gamma (\kappa + 1)^{1/2}\,\mathrm{d}s$, acting on the space of curves with pinned ends and given first order boundary data, Ω_{pq}. That is, Ω_{pq} is the space of regular curves in $\mathbf{S}^2(1)$ satisfying $\gamma(0) = p$, $\gamma(1) = q$, $(\mathrm{d}\gamma/\mathrm{d}t)(0) = e_1$, $(\mathrm{d}\gamma/\mathrm{d}t)(1) = e_2$, where $p, q \in \mathbf{S}^2(1)$ and $e_1, e_2 \in \mathbf{TS}^2$, are two fixed points of and tangent vectors to $\mathbf{S}^2(1)$, respectively. Then, computations of Sections 2 and 3 are basically the same and then a critical point of $\mathcal{F}(\gamma) : \Omega_{pq} \to \mathbf{R}$, is characterized by the Euler–Lagrange equation (4) which in this case reduces to

$$\kappa_{ss}(\kappa^2 + 1) = 3\kappa\kappa_s. \tag{39}$$

If we assume that $\kappa_s \neq 0$, then (39) can be easily integrated to

$$\kappa_s^2 = A^2(\kappa^2 + 1)^3, \tag{40}$$

where $A^2 \in (0, \infty)$. This equation leads in turn to

$$\kappa^2(s) = \frac{(As + B)^2}{1 - (As + B)^2} \tag{41}$$

with $B \in \mathbf{R}$. Let γ be a critical point corresponding to (41). From Proposition 2 and (40), $\mathcal{J} = -(\kappa^2 + 1)^{-1/2}T + AN$, is a Killing vector on γ, and we may choose geographical coordinates $x(\theta, \psi)$ so that, $x_\theta = b\mathcal{J}$. Moreover, since $\mathrm{d}|\mathcal{J}|^2/\mathrm{d}s = -2\kappa\kappa_s/(\kappa^2 + 1)^2$, and $\mathrm{d}^2|\mathcal{J}|^2/\mathrm{d}s^2|_{\kappa=0} = -2\kappa_s^2 < 0$, we have that the only zero of κ is a maximum of $|\mathcal{J}|^2$ and therefore it is the point where γ crosses the equator. On the other hand, if we denote by $\bar{\kappa}$ and τ the curvature and torsion of γ in $\mathbf{R}^3$, then

$$\bar{\kappa}(s) = (1 - (As + B)^2)^{-1/2}, \qquad \tau(s) = A(1 - (As + B)^2)^{-1/2}$$

and then $\tau(s)/\bar{\kappa}(s) = A$, what means that γ is a generalized helix of $\mathbf{R}^3$ contained in $\mathbf{S}^2(1)$. *Hence the solutions to the variational problem $\mathcal{F}(\gamma) : \Omega_{pq} \to \mathbf{R}$ are spherical generalized helices*. This result was already obtained by Santalo [20] by using a different approach, Fig. 4.

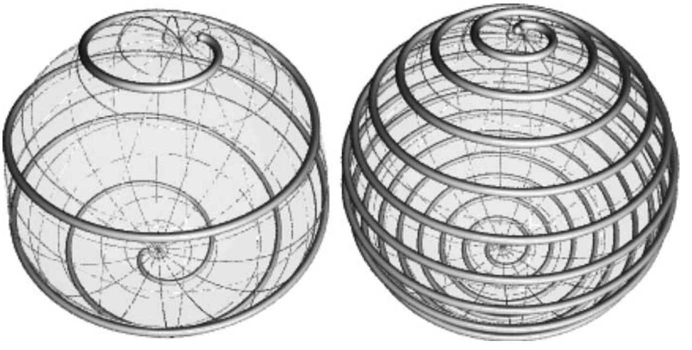

Fig. 4. Non-closed curves critical points of $\int_\gamma(\kappa+1)^{1/2}\,\mathrm{d}s$.

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